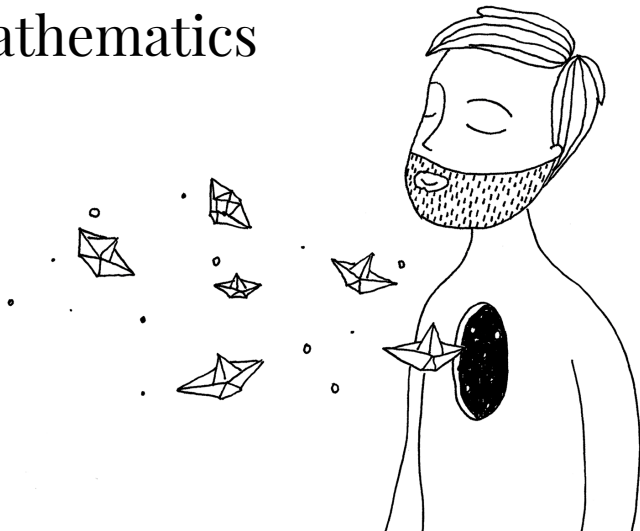


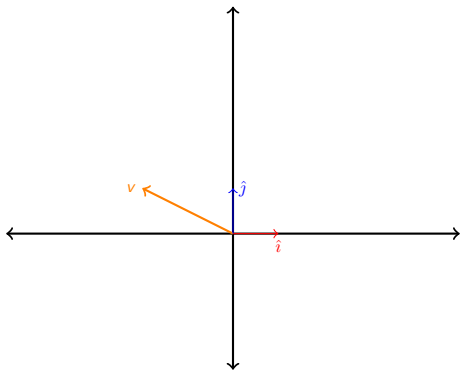
# 4509 – Bridging Mathematics

Matrices

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# Introduction



$$v = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$v = -2 \times \hat{i} + 1 \times \hat{j} = -2 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

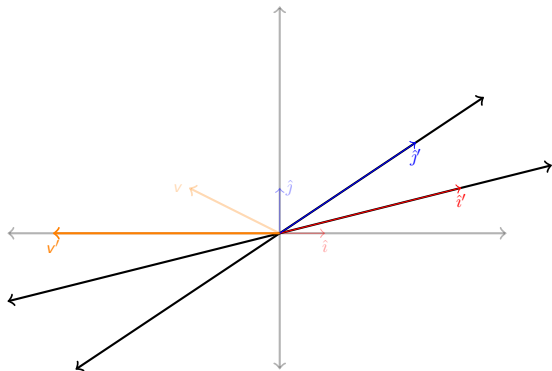
# Introduction

Being not very rigorous, we can define a linear transformation as a transformation on every vector on the plane that must satisfy 2 things:

1. Lines must be transformed into lines
2. The origin must remain in the same place

We will deal with the formal definition and rigor later...

# Introduction



$$\hat{i}' = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \hat{j}' = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$v = -2 \times \hat{i}' + 1 \times \hat{j}' = -2 \times \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$$

$$w = \begin{bmatrix} x \\ y \end{bmatrix} \text{ lands on } x \times \begin{bmatrix} 4 \\ 1 \end{bmatrix} + y \times \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4x + 3y \\ 1x + 2y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = x \times \hat{i} + y \times \hat{j}$$

$$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x + 3y \\ 1x + 2y \end{bmatrix} = x \times \hat{i}' + y \times \hat{j}'$$

# Introduction

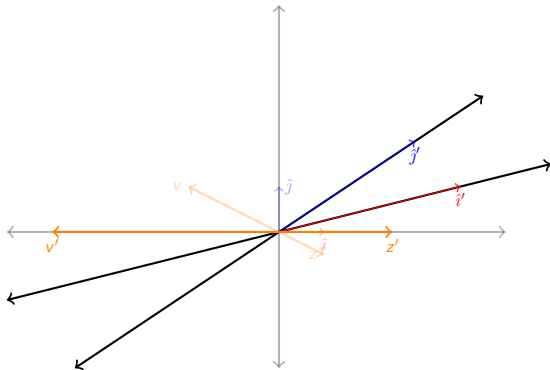
What about another vector in the same “direction” than  $v$ ? say

$$z = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

...

$$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \end{bmatrix}$$

# Introduction



# Introduction

So transforming two vectors in the same line, they both end up also in the same line...  
keep this in mind.

# Introduction

Could we take back  $\hat{i}'$  to  $\hat{i}$  and  $\hat{j}'$  to  $\hat{j}$ ?

Sure, if we apply the transform

$$\begin{bmatrix} 0.4 & -0.6 \\ -0.2 & 0.8 \end{bmatrix}$$

$$\begin{bmatrix} 0.4 & -0.6 \\ -0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Applying the inverse!



# Introduction

The important thing is that: if the vectors are linearly dependent, then we cannot invert the matrix, we just saw that two vectors that reside on the same line, end up in the same (although probably a different one) line.

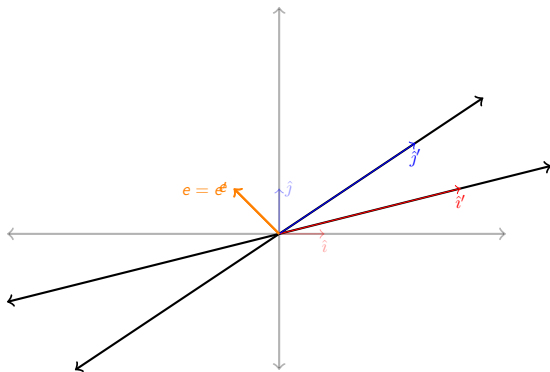
# Introduction

There are a couple of interesting vectors on the whole space when we apply this linear transformation...

Take for example the following vector:  $e = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

# Introduction



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

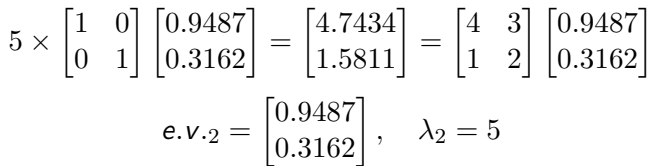
$$e.v._1 = [-1, 1], \quad \lambda_1 = 1$$

# Introduction

Or the vector:  $f = \begin{bmatrix} 0.9487 \\ 0.3162 \end{bmatrix}$

$$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.9487 \\ 0.3162 \end{bmatrix} = \begin{bmatrix} 4.7434 \\ 1.5811 \end{bmatrix}$$

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# Introduction

What happens with these vectors?

|              | $ev_1$                                  | $ev_2$  |
|--------------|---|---|
| $A \times$   | $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ | $\begin{bmatrix} 4.7434 \\ 1.5811 \end{bmatrix}$  |
| $A^2 \times$ | $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ | $\begin{bmatrix} 23.7171 \\ 7.9057 \end{bmatrix}$ |
| $A^3 \times$ | $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ | $\begin{bmatrix} 118.585 \\ 39.528 \end{bmatrix}$ |
| $\lambda$    | 1                                       | 5   |

## Definition

A real **matrix** is a rectangular array of real numbers.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{nm} \end{pmatrix}$$

Where  $a_{ij} \in \mathbb{R}$ .  $A$  is said to be an element of  $\mathbb{R}^{m \times n}$

A vector would be then a matrix with only 1 column!

Let  $A, B \in \mathbb{R}^{m \times n}$ . Let  $C \in \mathbb{R}^{n \times l}$ . Finally, let  $\alpha \in \mathbb{R}$ .

1.  $[A + B]_{ij} = a_{ij} + b_{ij}$
2.  $[A \cdot C]_{ik} = \sum_{j=1}^n a_{ij} \cdot c_{jk}$ , and it has a dimension  $m \times l$
3.  $[\alpha A]_{ij} = \alpha a_{ij}$



## Definition

Let  $A \in \mathbb{R}^{m \times n}$ ,  $A$ 's **transpose**, denoted  $A^t \in \mathbb{R}^{n \times m}$  is such that its elements are:

$$a_{ij}^t = a_{ji}$$

## Definition

Matrix  $A \in \mathbb{R}^{m \times n}$  is said to be **squared** if  $n = m$

## Definition

Matrix  $A$  is said to be **symmetric** if  $A^t = A$

## Definition

Matrix  $A$  is said to be **antisymmetric** if  $A^t = -A$

## Definition

The **Identity** is a squared matrix  $I_n \in \mathbb{R}^{n \times n}$  that has  $I_{ij} = 0$  if  $i \neq j$ , and  $I_{ij} = 1$  if  $i = j$ .

The identity has a nice property:  $AI_n = I_m A = A$  for any  $A \in \mathbb{R}^{m \times n}$ .

## Definition

Matrix  $A$  is **invertible**, if there is another matrix  $A^{-1}$  such that  $A \cdot A^{-1} = A^{-1} \cdot A = I$

## Conjecture

Given  $A, B, C \in \mathbb{R}^{n \times n}$

1.  $A + B = B + A$
2.  $A(BC) = (AB)C$
3.  $A(B + C) = AB + AC$
4.  $(A + B)^t = A^t + B^t$
5.  $(AB)^t = B^t A^t$
6.  $(A^t)^t = A$
7. *If  $A$  and  $B$  are invertible, then  $AB$  and  $BA$  are invertible as well. Furthermore*  
 $(AB)^{-1} = B^{-1}A^{-1}$
8. *If  $A$  is invertible, then*  $(A^t)^{-1} = (A^{-1})^t$

Quick quiz, 15 min, prove points 7 and 8. You can use points 1-6 as true and given.

# Solution

- 7 Start with  $AB$ , multiply by  $A^{-1}$  from the left, you are left with  $A^{-1}AB = IdB = B$ . Now multiply by  $B^{-1}$ , so you get  $B^{-1}A^{-1}AB = B^{-1}IdB = B^{-1}B = Id$ . Then  $(B^{-1}A^{-1})(AB) = Id$  so it must be that  $B^{-1}A^{-1} = (AB)^{-1}$ . To complete the proof, you need to show that you can do the same from the “right”.
- 8 Start with  $(A^{-1}A)^t = Id^t = Id$ , use property 5 and you get  $(A^{-1}A)^t = A^t(A^{-1})^t = Id$ , then  $(A^{-1})^t$  must be the inverse (again the only thing that is missing is to show that it works if you start with  $(AA^{-1})^t$  as well which is trivial).

## Conjecture

*The set of the matrices in  $\mathbb{R}^{m \times n}$ , together with the sum and scalar multiplication is a vector space.*

## Conjecture

*A squared matrix  $A \in \mathbb{R}^{n \times n}$  is invertible if and only if all of its columns are linearly independent.*

## Definition

Matrix  $A \in \mathbb{R}^{m \times n}$  is **upper triangular** if it has the following shape:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2m} & \dots & a_{2n} \\ 0 & 0 & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & a_{mm} & \dots & a_{mn} \end{bmatrix}$$

That is, it has zeroes below its main diagonal.

## Conjecture

*The set of the upper triangular matrices in  $\mathbb{R}^{n \times m}$ , with the sum and scalar multiplication is a vector subspace of  $\mathbb{R}^{n \times m}$ .*

## Definition

Matrix  $A$  is **lower triangular** if  $A^t$  is upper triangular.

## Definition

Matrix  $A$  is **diagonal** if it is upper and lower triangular at the same time.

## Definition

The **rank** of a matrix  $A$ , denoted by  $rank(A)$  is the maximum number of linearly independent rows or columns of  $A$ .



A convenient way to write down a system of equations:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

It would be  $AX = B$ , where,

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

## Definition

Given a system of equations  $AX = B$ ,

- $\hat{X} \in \mathbb{R}^n$  is a **particular solution** of the system if  $A\hat{X} = B$ .
- $X_0$  is an **homogeneous solution** if  $AX_0 = 0$ .

Note that for  $\lambda \in \mathbb{R}$ ,  $A(\hat{X} + \lambda X_0) = A\hat{X} + \lambda AX_0 = B + 0 = B$ .

## Definition

The **kernel** of  $A \in \mathbb{R}^{m \times n}$  is defined as:

$$\text{Ker}(A) := \{X \in \mathbb{R}^n | AX = 0\}$$

## Conjecture

$\text{Ker}(A) \subseteq \mathbb{R}^n$  is a vector subspace of  $\mathbb{R}^n$

## Definition

The dimension of  $\text{Ker}(A)$  is called the **nullity** — or nullspace —, and it is denoted by  $\text{Null}(A)$ . If  $\text{Ker}(A) = \{0\}$ , then  $\text{Null}(A) = 0$ .

Note that a system of equations as the one shown before, has unique solution only if  $\text{Null}(A) = 0$ .

## Conjecture

*Matrix  $A \in \mathbb{R}^{n \times n}$  is invertible if and only if  $\text{Null}(A) = 0$ .*

## Conjecture

*$A \in \mathbb{R}^{n \times n}$  is invertible if and only if the system  $AX = B$  has a unique solution, for any  $B \in \mathbb{R}^n$ .*

## Definition

The **image** of  $A \in \mathbb{R}^{n \times n}$  is defined as:

$$Im(A) := \{Y \in \mathbb{R}^n | \exists X \in \mathbb{R}^n, Y = AX\} \equiv \{AX | X \in \mathbb{R}^n\}$$

## Conjecture

*Let  $A \in \mathbb{R}^{n \times n}$ .  $Im(A)$  is a vector subspace of  $\mathbb{R}^n$ .*

## Definition

The dimension of  $Im(A)$  is called the **range** of  $A$ . Let's denote it as  $R(A)$ .

## Conjecture

*Let  $A \in \mathbb{R}^{n \times n}$ .  $R(A)$  is the number of l.i. columns of  $A$ .*

## Conjecture

*Consider  $A \in \mathbb{R}^{n \times n}$ . It holds that  $\text{Null}(A) + R(A) = n$ .*

## Definition

The quadratic form associated to  $A$  is a function  $Q_A : \mathbb{R}^n \rightarrow \mathbb{R}$  such that for any  $X \in \mathbb{R}^n$ ,

$$Q_A(X) = X^t A X \in \mathbb{R}$$

## Conjecture

*For any matrix  $A \in \mathbb{R}^{n \times n}$ , there are always symmetric and antisymmetric matrices  $S$  and  $T$  such that*

$$A = S + T$$

Note: Let  $S = \frac{A+A^t}{2}$  and  $T = \frac{A-A^t}{2}$ . While  $S$  is symmetric,  $T$  is antisymmetric,

## Corollary

*A quadratic form can be represented as*

$$Q_A(X) = X^t S X$$

*with  $S$  symmetric.*



Quick quiz! 15 min to prove the corollary.

# Solution

- Let  $A = (S + T)$
- Then  $Q_A(X) = X^t(S + T)X = X^tSX + X^tTX$
- But  $X^tTX \in \mathbb{R}$ , so  $(X^tTX)^t = X^tTX$  (a number trasposed is the same number).
- So you end up that  $X^tTX = (X^tTX)^t = X^tT^tX$
- But  $T$  is anytsymmetric so  $T^t = -T...$
- Then  $X^tTX = -X^tTX$ , so if  $X^tTX$  is the number  $z$ , you have  $z = -z$ , that only is true for  $z = 0$ .
- Then  $Q_A(X) = X^tSX$

## Definition

Let  $A \in \mathbb{R}^{n \times n}$ , symmetric. Consider the quadratic form  $Q_A(X) = X^t A X$ . If for any  $X \in \mathbb{R}^n \setminus \{0\}$ ,

1.  $Q_A(X) > 0$ ,  $A$  is **positive definite**,
2.  $Q_A(X) \geq 0$ ,  $A$  is **positive semi-definite**,
3.  $Q_A(X) < 0$ ,  $A$  is **negative definite**,
4.  $Q_A(X) \leq 0$ ,  $A$  is **negative semi-definite**.

## Definition

$\lambda \in \mathbb{C}$  is an **eigenvalue** (or characteristic value) of matrix  $A \in \mathbb{R}^{n \times n}$  if there is a vector, called **eigenvector**,  $X_\lambda \in \mathbb{R}^n \setminus \{0\}$  such that

$$AX_\lambda = \lambda X_\lambda$$

## Conjecture

*Let  $A \in \mathbb{R}^{n \times n}$ , and  $\lambda_1$  and  $\lambda_2$  two eigenvalues of  $A$ , with  $\lambda_1 \neq \lambda_2$ . If  $V_1$  is the vector subspace associated to  $\lambda_1$ , and  $V_2$  is the vector subspace of  $\lambda_2$  then  $V_1$  and  $V_2$  are linearly independent.*

## Conjecture

*Given  $A \in \mathbb{R}^{n \times n}$  symmetric, then its eigenvalues are real valued.*

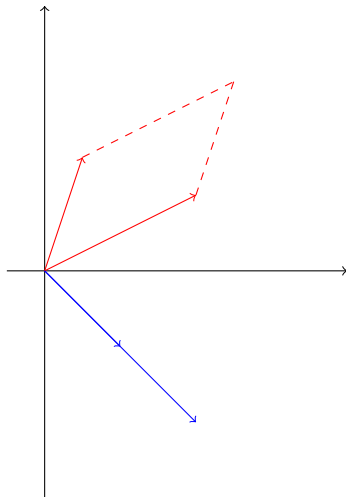
## Definition

The **determinant** of a squared matrix  $A$  is the hyper-volume of the figure formed by the column vectors of the matrix.

## Example

Consider the matrices,

$$A = \begin{pmatrix} 1/2 & 2 \\ 3/2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$$



It is easy to see that, given our definition,  $\det(B) = 0$ . It is also easy to show that  $\det(A) = |1/2 \times 1 - 3/2 \times 2| = 5/2$ .

How to calculate the determinant of a big matrix? Recursively. Let  $A \in \mathbb{R}^{n \times n}$ . Define  $A_{ij}$  as:

$$A_{ij} = \begin{pmatrix} a_{11} & \dots & a_{1,j-1} & a_{1,j+1} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{n,j-1} & a_{n,j+1} & \dots & a_{nn} \end{pmatrix}$$

That is, what is left of  $A$  after removing row  $i$  and column  $j$ .



Then,

$$\det(A) = \sum_{k=1}^n (-1)^{i+k} a_{ik} \det(A_{ik})$$

You can choose any  $i$  that you prefer.

## Conjecture

- *A squared matrix is invertible if and only if its determinant is different from zero.*
- *Take a finite set of matrices  $\mathbb{A} \subseteq \mathbb{R}^{n \times n}$ , with  $A_i$  being the  $i$ th element of  $\mathbb{A}$  then,*

$$\det(A_1 A_2 \dots A_k) = \det(A_1) \det(A_2) \dots \det(A_k)$$

- *If  $A$  is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

- *For any squared  $A$  it holds that  $\det(A^t) = \det(A)$ .*

Note that  $\lambda$  is an eigenvalue if

$$AX_\lambda = \lambda X_\lambda \quad \text{with} \quad X_\lambda \neq 0$$

so  $\lambda$  is an eigenvalue of  $A$  if

$$(A - \lambda I)X_\lambda = 0$$

or  $X_\lambda \in \ker(A - \lambda I)$ , which implies that  $\ker(A - \lambda I) \neq \{0\}$ , and therefore  $(A - \lambda I)$  must be not invertible! But if  $(A - \lambda I)$  is not invertible, then  $\det(A - \lambda I) = 0$ .

### Corollary

*$\lambda$  is an eigenvalue of  $A$  if and only if  $\det(A - \lambda I) = 0$ .*

## Definition

Given  $A \in \mathbb{R}^{n \times n}$ , the **characteristic polynomial** of  $A$  is defined as the function  $p_A : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$p_A(\lambda) = \det[A - \lambda I]$$

So,  $\lambda$  is an eigenvalue of  $A$  if  $p_A(\lambda) = 0$

## Conjecture

*If  $A$  is symmetric, then the eigenvectors of different eigenvalues are orthogonal.*

For practical reasons, consider the matrix  $V$  as the matrix that has in its columns the eigenvectors of  $A$ , and  $D(\lambda)$  the diagonal matrix that contains in the column  $i$ , the eigenvalue that corresponds to the eigenvector in the column  $i$  in  $V$ .

Note that:

$$AV = VD(\lambda) \quad \Leftrightarrow \quad A = VD(\lambda)V^{-1}$$

Note that given the properties of matrix multiplication

$$A^{-1} = VD\left(\frac{1}{\lambda}\right)V^{-1}$$

which is one of the fundamental properties of the symmetric matrices.

## Conjecture

*Given  $A \in \mathbb{R}^{n \times n}$ , symmetric. It holds that,*

$$A = VDV^t$$

*With  $D$  the diagonal with the eigenvalues of  $A$  and  $V$  the unit eigenvectors of  $A$ .*

## Conjecture

*Let  $A \in \mathbb{R}^{n \times n}$ , symmetric.*

- 1.  $A$  is positive definite if all the eigenvalues of  $A$  are strictly positive.*
- 2.  $A$  is positive semi definite if all the eigenvalues are nonnegative.*
- 3.  $A$  is negative definite if all the eigenvalues are strictly negative.*
- 4.  $A$  is negative semidefinite if all the eigenvalues are non positive.*

## Definition

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear if, for any  $X, Y \in \mathbb{R}^n$ , and for any  $\alpha \in \mathbb{R}$

$$f(X + Y) = f(X) + f(Y), \quad f(\alpha X) = \alpha f(X)$$



## Definition

The **trace** of a square matrix  $A$  ( $tr(A)$ ) is the sum of the elements of its diagonal.

## Theorem

Let  $A \in \mathbb{R}^{n \times n}$ , then:

- the product of the eigenvalues of  $A$  is equal to its determinant, that is,

$$\det(A) = \prod_{i=1}^n \lambda_i$$

- the sum of the eigenvalues of  $A$  is equal to its trace, that is,

$$\sum_{i=1}^n a_{i,i} = \sum_{i=1}^n \lambda_i$$

- if  $A$  is a triangular matrix, then its eigenvalues are the coefficients in the principal diagonal of the matrix, i.e.,

$$\lambda_i = a_{i,i}$$